12 Electromagnetic Waves With $\mathbf{B}_0 = 0$

A brief review of light waves in a vacuum ($j = 0$):

$$\begin{aligned} \nabla \times \mathbf{E}_1 &= -\frac{\partial \mathbf{B}_1}{\partial t} \\ c^2 \nabla \times \mathbf{B}_1 &= \frac{\partial \mathbf{E}_1}{\partial t} \end{aligned} \tag{5}$$

where $\varepsilon_0 \mu_0 = c^2$

$$c^2 \nabla \times (\nabla \times \mathbf{B}_1) = \nabla \times \frac{\partial \mathbf{E}_1}{\partial t} = -\frac{\partial^2}{\partial t^2} \mathbf{B}_1$$

Again assuming planes waves varying as $exp[i(kx - \omega t)]$, we have

$$\omega^2 \mathbf{B}_1 = -c^2 \mathbf{k} \times (\mathbf{k} \times \mathbf{B}_1) = -c^2 [\mathbf{k}(\mathbf{k} \cdot \mathbf{B}_1) - k^2 \mathbf{B}_1]$$

$$\omega^2 = k^2 c^2$$

c is the phase velocity ω/k of light waves.

In a plasma with $\mathbf{B}_0 = 0$, we must add a term $\mathbf{j}_1/\epsilon_0$ to account for currents due to first-order charged particle motions:

$$c^2 \nabla \times \mathbf{B}_1 = \frac{\mathbf{j}_1}{\epsilon_0} + \frac{\partial \mathbf{E}_1}{\partial t}$$

The time derivative of this is

$$c^2 \nabla \times \dot{\mathbf{B}}_1 = \frac{1}{\epsilon_0} \frac{\partial \mathbf{j}_1}{\partial t} + \frac{\partial^2 \mathbf{E}_1}{\partial t^2}$$

where the curl of Eq. 5 is

$$\nabla \times \nabla \times \mathbf{E}_1 = \nabla(\nabla \cdot \mathbf{E}_1) - \nabla^2 \mathbf{E}_1 = -\nabla \times \dot{\mathbf{B}}_1$$

Eliminating $\nabla \times \dot{\mathbf{B}}_1$ and assuming an $exp[i(kx - \omega t)]$:

$$-\mathbf{k}(\mathbf{k} \cdot \mathbf{E}_1) + k^2 \mathbf{E}_1 = \frac{i\omega}{\epsilon_0 c^2} \mathbf{j}_1 + \frac{\omega^2}{c^2} \mathbf{E}_1$$

By transverse waves we mean, $\mathbf{k} \cdot \mathbf{E}_1 = 0$

$$(\omega^2 - c^2 k^2) \mathbf{E}_1 = -i\omega \mathbf{j}_1/\epsilon_0 \tag{6}$$

If we consider light waves or microwaves, these will be of such high frequency that the ions can be considered as fixed. The current $\mathbf{j}_1$ then comes entirely from electron motion

$$\mathbf{j}_1 = -n_0 e \boldsymbol{v}_{e1}$$

From the linearized electron equation of motion, we have (for $T_e = 0$)

$$\begin{aligned} m\frac{\partial \boldsymbol{v}_{e1}}{\partial t} &= -e\mathbf{E}_1 \\ \boldsymbol{v}_{e1} &= \frac{e\mathbf{E}_1}{im\omega} \end{aligned}$$

$\Rightarrow$

$$(\omega^2 - c^2k^2)\mathbf{E}_1 = \frac{n_0 e^2}{\epsilon_0 m}\mathbf{E}_1$$

$$\omega^2 = \omega_p^2 + k^2c^2$$

This is the dispersion relation for electromagnetic waves propagating in a plasma with no dc magnetic field.

Note:

1. $v_\phi > c$,but $v_g < c$
2. at large kc, Electromagnetic waves become ordinary light waves.
3. *Cutoff*: if $\omega < \omega_p$, k is imaginary, the wave ($\tilde{}\, exp(ikx)$), will be exponentially attenuated, the wave cannot propagate.
4. The index of refraction $N \equiv \frac{c}{v_\phi} = \frac{kc}{\omega} < 1$, A convex plasma is divergent rather than convergent – inportant in the lasersolenoid proposal for a linear fusion reactor.
5. Best know effect of the plasma cutoff is the application to short wave radio commuincation. When a radio wave reachs an altitude in the ionospehre where the plasma density is sufficiently high, the wave is reflected, making it possible to send signals around the earth. To communicate with space vehicles, it is necessary to use frequencies above the critcial frequency (for $n\tilde{}10^{12}m^{-3,}f_c\tilde{}10\ MHz$) in order to penetrate the ionosphere. However, during reentry of a space vehicle, a plasma is generated by the intense heat of friction. This causes a plasma cutoff, resulting in a communications blackout during reentry.

13 Electromagnetic Waves Perpendicular to $\mathbf{B}_0$

We now consider the propagation of electromagnetic waves when a magnetic field is present. We treat first the case of perpendicular propagation, $\mathbf{k} \perp \mathbf{B}_0$. If we take transverse waves, with $\mathbf{k} \perp \mathbf{E}_1$, there are still two choice: $\mathbf{E}_1$ can be parallel to $\mathbf{B}_0$ or perpendicular to $\mathbf{B}_0$.

13.1 Ordinary Waves, $\mathbf{E}_1 // \mathbf{B}_0$

Taking $\mathbf{B}_0 = B_0\hat{z}, \mathbf{E}_1 = E_1\hat{z}$,and $\mathbf{k} = k\hat{x}$

The wave equation for this case is still given by Eq 6

$$(\omega^2 - c^2k^2)\mathbf{E}_1 = -i\omega\mathbf{j}_1/\epsilon_0 = in_0e\omega\boldsymbol{v}_{e1}/\epsilon_0$$

Since $\mathbf{E}_1 = E_1\hat{z}$, we need only the component v_{ez} :

$$m\frac{\partial v_{ez}}{\partial t} = -eE_1$$

Since this is the same as the equation for $\mathbf{B}_0 = 0$, the result is the same

$$\omega^2 = \omega_p^2 + k^2c^2$$

This wave, with $\mathbf{E}_1 // \mathbf{B}_0$, is called the ordinary wave. (not affected by the magnetic field)

13.2 Extraordinary Wave $\mathbf{E}_1 \perp \mathbf{B}_0$

Waves with $\mathbf{E}_1 \perp \mathbf{B}_0$ tend to develope a component E *along* $\mathbf{k}$, thus becoming partly longitudinal and partly transverse. We must allow $\mathbf{E}_1$ to have both x and y components.

$$\mathbf{E}_1 = E_x\hat{x} + E_y\hat{y}$$

The linearized electron equation of motion (with $kT_e = 0$) is now

$$-im\omega\boldsymbol{v}_{e1} = -e(\mathbf{E}_1 + \boldsymbol{v}_{e1} \times \mathbf{B}_0)$$

$$\begin{aligned} v_x &= \frac{-ie}{m\omega}(E_x + v_yB_0) \\ v_y &= \frac{-ie}{m\omega}(E_y - v_xB_0) \end{aligned}$$

The subscripts 1 and e have been supressed.

$$\begin{aligned} v_x &= \frac{e}{m\omega}(-iE_x - \frac{\omega_c}{\omega}E_y)(1 - \frac{\omega_c^2}{\omega^2})^{-1} \\ v_y &= \frac{e}{m\omega}(-iE_y + \frac{\omega_c}{\omega}E_x)(1 - \frac{\omega_c^2}{\omega^2})^{-1} \end{aligned}$$

The wave equation:

$$(\omega^2 - c^2k^2)\mathbf{E}_1 + c^2kE_x\mathbf{k} = -i\omega\mathbf{j}_1/\epsilon_0 = in_0e\omega\boldsymbol{v}_{e1}/\epsilon_0 \tag{7}$$

x and y components:

$$\begin{aligned}
\omega^2 E_x &= -\frac{i\omega n_{0e}}{\epsilon_0}(iE_x + \frac{\omega_c}{\omega}E_y)(1-\frac{\omega_c^2}{\omega^2})^{-1} \\
(\omega^2 - c^2k^2)E_y &= -\frac{i\omega n_{0e}}{\epsilon_0}(iE_y - \frac{\omega_c}{\omega}E_x)(1-\frac{\omega_c^2}{\omega^2})^{-1}
\end{aligned}$$

or

$$\begin{aligned}
[\omega^2(1-\frac{\omega_c^2}{\omega^2}) - \omega_p^2]E_x + i\frac{\omega_p^2\omega_c}{\omega}E_y &= 0 \\
[(\omega^2 - c^2k^2)(1-\frac{\omega_c^2}{\omega^2}) - \omega_p^2]E_y - i\frac{\omega_p^2\omega_c}{\omega}E_x &= 0
\end{aligned}$$

These are two simultaneous equation for E$_x$ and E$_y$ which are compatible only if the determinant of the coefficients vanishs, i.e.

$$(\omega^2 - \omega_h^2)[\omega^2 - \omega_h^2 - c^2k^2(1-\frac{\omega_c^2}{\omega^2})] = (\frac{\omega_p^2\omega_c}{\omega})^2$$

$\Rightarrow$

$$\frac{c^2k^2}{\omega^2} = \frac{c^2}{v_\phi^2} = 1 - \frac{\omega_p^2}{\omega^2}\frac{\omega^2-\omega_p^2}{\omega^2-\omega_h^2} \tag{8}$$

This is the dispersion relation for the extraodinary wave. It is an electromagnetic wave, partly transverse and partly longitudianl, which propagates perpendicular to $\mathbf{B}_0$ with $\mathbf{E}_1$ perpendicular to $\mathbf{B}_0$.

14 Cutoffs and Resonance

cutoff: the index of refraction $N = ck/\omega$ goes to zero or the wavelength becomes infinite.

resonance: the index of refraction goes to infinite or the wavelength becomes zero.

The resonance of the extraordinary wave is found by setting k equal to infinity in Eq. 8. So that

$$\omega^2 = \omega_h^2 = \omega_p^2 + \omega_c^2$$

At resonance the extraordinary wave loses its electromagnetic character and becomes an electrostatic oscillation.

The cutoffs of the extraorinary wave are found by setting k equal to 0 in Eq. 8.

$$1 = \frac{\omega_p^2}{\omega^2}\frac{\omega^2-\omega_p^2}{\omega^2-\omega_h^2}$$

$\Longrightarrow$

$$
\begin{aligned}
\omega_R &= \frac{1}{2}[\omega_c + (\omega_c^2 + 4\omega_p^2)^{1/2}] \qquad \text{right-hand cutoff} \\
\omega_L &= \frac{1}{2}[-\omega_c + (\omega_c^2 + 4\omega_p^2)^{1/2}] \qquad \text{left-hand cutoff}
\end{aligned}
$$

The cutoff and resonance frequencies divide the dispersion diagram into regions of propagation and nonpropagation. Instead of the ususal $\omega - k$ diagram, it is more enlightening to give a plot of phase velocity versus frequency.

15 Electromagnetic Waves Parallel to $\mathbf{B}_0$

$$\mathbf{k} = k\hat{z} \quad \mathbf{E}_1 = E_x\hat{x} + E_y\hat{y}$$

The wave equation 7 can still be used if we simply change $\mathbf{k}$ from $k\hat{x}$ to $k\hat{z}$, the components are now

$$
\begin{aligned}
(\omega^2 - c^2k^2)E_x &= \frac{\omega_p^2}{1 - \omega_c^2/\omega^2}(E_x - \frac{i\omega_c}{\omega}E_y) \\
(\omega^2 - c^2k^2)E_y &= \frac{\omega_p^2}{1 - \omega_c^2/\omega^2}(E_y + \frac{i\omega_c}{\omega}E_y)
\end{aligned}
$$

using the abbreviation:

$$\alpha \equiv \frac{\omega_p^2}{1 - \omega_c^2/\omega^2}$$

$$
\begin{aligned}
(\omega^2 - c^2k^2 - \alpha)E_x + i\alpha\frac{\omega_c}{\omega}E_y &= 0 \\
(\omega^2 - c^2k^2 - \alpha)E_y - i\alpha\frac{\omega_c}{\omega}E_x &= 0
\end{aligned}
$$

Setting the determinant of the coefficients to zero, we have

$$
\begin{aligned}
(\omega^2 - c^2k^2 - \alpha)^2 &= (\alpha\frac{\omega_c}{\omega})^2 \\
&\Rightarrow \quad \omega^2 - c^2k^2 = \frac{\omega_p^2}{1 \mp (\omega_c/\omega)}
\end{aligned}
$$

The dispersion relation are

$$
\begin{aligned}
N^2 &= \frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2}{1 - (\omega_c/\omega)} \quad \text{(R wave)} \\
N^2 &= \frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2}{1 + (\omega_c/\omega)} \quad \text{(L wave)}
\end{aligned}
$$

The R and L waves turn out to be circularly polarized, the designations R and L meaning, respectly, right-hand circular polarization and left-hand circular polarization. The electric field vector for the R wave rotates clockwise in time as viewed along the direction of B_0, and vice versa for the L wave.

Cutoffs and resonances of the R/L waves:

Resonances: for the R-wave, $\omega = \omega_c$; The L wave does not have resonance for positive ω. If we had included ion motions in our treatment, the L wave would have been found to have a resonance at $\omega = \Omega_c$.

Cutoffs: for the R-wave, $\omega = \omega_R$; for the L wave, has a lower cutoff frequency $\omega_{L.}$ This is the reason for the notation ω_R and ω_L chosen previously.

Whistler mode: The R-wave in the low-frequency region is called the whistler mode and is of extreme importance in the study of ionspheric phenomena.

To summarize: the principal electromagnetic waves propagating along B_0 are a right-hand (R) or a left-hand (L) circularly polarized wave; the principal waves propagating across B_0 are a plane-polarized wave (O-wave) and an elliptically polarized wave (X-wave).

16 Experimental Consequences

16.1 The Whistler Mode

Early investigators of radio emissions from the ionosphere were rewarded by various whistling sounds in the audiofrequency range. There is typically a series of decending glide tones, which can be heard over a loundspeaker.

A lighting flash occurs in the Southern Hemisphere –> radio noise of all frequencies is generated –> R waves traveling along the earth's magnetic field lines –> different frequencies arrive at different times, the low frequencies arrive later (why?) –> the decending tone. (Either the whistles lie directly in the audio range or they can easily be converted into audio signals by heterodyning).

16.2 Faraday Rotation

A plane-polarized wave sent along a magnetic field in a plasma will suffer a rotation of its plane of polarization. This can be understand stood in terms of the difference in phase velocity of the R and L waves.

Consider the plane-polarized wave to be the sum of an R-wave and an L wave (at the same frequency, of course), but for large ω, the phase velocity of the R-wave is faster than the L wave. On propagating a distance ds, the phases of the two modes are shifted in relation to each other by

$$d\psi = (k_L - k_R)ds = (\frac{1}{v_{ph}^L} - \frac{1}{v_{ph}^R})\omega ds$$

The plane of polarization is seen to have rotated.

Faraday rotation –> ω_p –> density

17 Hydromagnetic Waves

Consider low-frequency ion oscillations in the presence of a magnetic field.

Alfven Wave: the hydromagnetic wave along B_0

Magnetosonic wave: across B_0

The Alfven wave in plane geometry has $\mathbf{k}$ along $\mathbf{B}_0$, $\mathbf{E}_1$ and $\mathbf{j}_1$ perpendicular to $\mathbf{B}_0$; and $\mathbf{B}_1$ and $\mathbf{v}_1$ perpendicular to both $\mathbf{B}_0$ and $\mathbf{E}_1$. From Maxwell's equation, we have, as ususal:

$$\nabla \times \nabla \times \mathbf{E}_1 = -\mathbf{k}(\mathbf{k} \cdot \mathbf{E}_1) + k^2 \mathbf{E}_1 = \frac{\omega^2}{c^2}\mathbf{E}_1 + \frac{i\omega}{\epsilon_0 c^2}\mathbf{j}_1$$

By assumption, $\mathbf{k} = \mathrm{k}\hat{z}$, $\mathbf{E}_1 = E_1 x$, only the x component of this equation is nontrivial. The current $\mathbf{j}_1$ now has contributions from both ions and electrons, since we are considering low frequencies.

$$\epsilon_0(\omega^2 - c^2k^2)E_1 = -i\omega n_0 e(\upsilon_{ix} - \upsilon_{ex})$$

Assuming $\mathrm{T}_i = 0$, the ion equation of motion obtained previously in Eq. 4:

$$\begin{aligned} \upsilon_{ix} &= \frac{ie}{M\omega}(1 - \frac{\Omega_c^2}{\omega^2})^{-1}E_1 \\ \upsilon_{iy} &= \frac{e}{M\omega}\frac{\Omega_c}{\omega}(1 - \frac{\Omega_c^2}{\omega^2})^{-1}E_1 \end{aligned}$$

The corresponding solution to the electron eqution of motion is found by letting M –>m, e –> -e, $\Omega_c \to \omega_c$, and taking the limit $\omega_c^2 >> \omega^2$:

$$\begin{aligned} \upsilon_{ex} &= \frac{ie}{m\omega}\frac{\omega^2}{\omega_c^2}E_1 \to 0 \\ \upsilon_{ey} &= -\frac{e}{m}\frac{\omega_c}{\omega^2}\frac{\omega^2}{\omega_c^2}E_1 = -\frac{E_1}{B_0} \end{aligned}$$

In this limit, the Larmor grarations of the electrons are neglected, and the electrons have simply an $\mathbf{E}\times\mathbf{B}$ drift in y direction.

$\Rightarrow$

$$\epsilon_0(\omega^2 - c^2k^2)E_1 = -i\omega n_0 e\frac{ie}{M\omega}(1 - \frac{\Omega_c^2}{\omega^2})^{-1}E_1$$

$$\omega^2 - c^2k^2 = \Omega_p^2(1 - \frac{\Omega_c^2}{\omega^2})^{-1}$$

Assuming $\omega^2 << \Omega_p^2$,hydromagnetic waves have frequencies well below ion cyclotron resonances. In this limit:

$$\omega^2 - c^2k^2 = -\omega^2 \frac{\Omega_p^2}{\Omega_c^2} = -\omega^2 \frac{n_0 e^2}{\epsilon_0 M} \frac{M^2}{e^2 B_0^2} = -\omega^2 \frac{\rho}{\epsilon_0 B_0^2}$$

$$\frac{\omega^2}{k^2} = \frac{c^2}{1 + (\rho/\epsilon_0 B_0^2)} = \frac{c^2}{1 + (\rho\mu_0/B_0^2)c^2} \tag{9}$$

This simply gives the phase velocity for an electromagnetic waves in a dielectric medium:

$$\frac{\omega}{k} = \frac{c}{(\epsilon_R \mu_R)^{1/2}} = \frac{c}{(\epsilon_R)^{1/2}} \quad \mu_R = 1$$

ϵ is much larger than unity for most cases, and Eq. 9 can be written approximately:

$$\frac{\omega}{k} = \upsilon_\phi = \frac{B_0}{(\mu_0 \rho)^{1/2}}$$

The hydromagnetic wave travel along B_0 at a constant velocity υ_A : *Alfven velocity*

This is a characteristic velocity at which perturbations of the lines of force travel. The dielectric constant can now be written:

$$\epsilon_R = \epsilon/\epsilon_0 = 1 + (c^2/\upsilon_A^2)$$

Physical Picture: The fluid and the field lines oscillate together as if the particles were plucked to the lines. The lines of force act as if they were mass-loaded strings under tension, and an Alfven wave can be regarded as the propagating distrubance occurring when the strings are plucked.

18 Magnetsonic Waves

Finally, we consider low-frequency electromagnetic waves propagating across B_0.

$\mathbf{B}_0 = B_0\hat{z}; \mathbf{E}_1 = E_1\hat{x}$, but we now let $\mathbf{k} = k\hat{y}$. Now we see the $\mathbf{E}_1 \times \mathbf{B}_0$ drifts along k, so the the plasma will be compressed and released in the course of the oscillation. It is necessary, therefore, to keep the ∇p term in the equation of motion. For the ions

$$Mn_0 \frac{\partial \boldsymbol{\upsilon}_{i1}}{\partial t} = en_0(\mathbf{E}_1 + \boldsymbol{\upsilon}_{i1} \times \mathbf{B}_0) - \gamma_i k_B T_i \nabla n_1$$

$\Rightarrow$

$$\begin{aligned} \upsilon_{ix} &= \frac{ie}{M\omega}(E_x + \upsilon_{iy}B_0) \\ \upsilon_{iy} &= \frac{ie}{M\omega}(-i\upsilon_{ix}B_0) + \frac{k}{\omega}\frac{\gamma_i k_B T_i}{M}\frac{n_1}{n_0} \end{aligned}$$

The equation of continuity yields:

$$\frac{n_1}{n_0} = \frac{k}{\omega}\upsilon_{iy}$$

so that

$$\upsilon_{iy} = -\frac{ie}{M\omega}\upsilon_{ix}B_0 + \frac{k^2}{\omega^2}\frac{\gamma_i k_B T_i}{M}\upsilon_{iy}$$

with

$$A = \frac{k^2}{\omega^2}\frac{\gamma_i k_B T_i}{M}$$

This becomes

$$\upsilon_{iy}(1-A) = -\frac{i\Omega_c}{\omega}\upsilon_{ix}$$

$\Rightarrow$

$$\begin{aligned}\upsilon_{ix} &= \frac{ie}{M\omega}E_x + \frac{i\Omega_c}{\omega}(-\frac{i\Omega_c}{\omega})(1-A)^{-1}\upsilon_{ix} \\ &\Longrightarrow \upsilon_{ix}(1-\frac{\Omega_c^2/\omega^2}{1-A}) = \frac{ie}{M\omega}E_x\end{aligned}$$

so that the wave equation:

$$\epsilon_0(\omega^2 - c^2k^2)E_x = -i\omega n_0 e(\upsilon_{ix} - \upsilon_{ex})$$

Take the limit of small electron mass, so that $\omega^2 << \omega_c^2$ and $\omega^2 << k^2\upsilon_{the}^2$:

$$\upsilon_{ex} = \frac{ie}{m\omega}\frac{\omega^2}{\omega_c^2}(1-\frac{k^2}{\omega^2}\frac{\gamma_e k_B T_e}{m})E_x \rightarrow -\frac{ik^2}{\omega B_0^2}\frac{\gamma_e k_B T_e}{e}E_x$$

Putting the last there equations together we have

$$\epsilon_0(\omega^2 - c^2k^2)E_x = -i\omega n_0 e[\frac{ie}{M\omega}E_x(\frac{1-A}{1-A-\Omega_c^2/\omega^2)}) + \frac{ik^2M}{\omega B_0^2}\frac{\gamma_e k_B T_e}{eM}E_x$$

We shall again assume $\omega^2 << \Omega_c^2$:

$$(\omega^2 - c^2k^2) = -\frac{\Omega_p^2}{\Omega_c^2}\omega^2(1-A) + \frac{k^2c^2}{\upsilon_A^2}\frac{\gamma_e k_B T_e}{M}$$

note $\Omega_p^2/\Omega_c^2 = c^2/\upsilon_A^2$:

$$\omega^2(1+\frac{c^2}{\upsilon_A^2}) = c^2k^2(1+\frac{\gamma_e k_B T_e + \gamma_i k_B T_i}{M\upsilon_A^2}) = c^2k^2(1+\frac{\upsilon_s^2}{\upsilon_A^2})$$

where υ_s is the acoustic speed. Finally we have

$$\frac{\omega^2}{k^2} = c^2\frac{\upsilon_s^2 + \upsilon_A^2}{c^2 + \upsilon_A^2}$$

This is the despersion relation for the magnetosonic wave propagating perpendicular to B_0.

Notes:

1. When $B_0 \to 0, v_A \to 0$,the magnetosonic wave truns into an ordinary ion acoustic wave.

2. In the limit T$\to 0, v_s \to 0$, the pressure gradient forces vanish, and the wave becomes a modified Alfven wave.

3. $v_\phi > v_A$ – "fast" hydromagnetic wave.

19 Summary of Elementary Plasma Waves

Electron waves (electrostatic)

$\mathbf{B}_0 = 0$ or $\mathbf{k} \parallel \mathbf{B}_{0:}$ $\omega^2 = \omega_p^2 + \frac{3}{2}k^2 v_{th}^2$ (plasma oscillations)

$\mathbf{k} \perp \mathbf{B}_{0:}$ $\omega^2 = \omega_p^2 + \omega_c^2$ (Upper hybrid oscillations)

Ion waves (electrostatic)

$\mathbf{B}_0 = 0$ or $\mathbf{k} \parallel \mathbf{B}_{0:}$ $\omega^2 = k^2 v_s^2 = k^2 \frac{\gamma_e k_B T_e + \gamma_i k_B T_i}{M}$ (Acoustic waves)

$\mathbf{k} \perp \mathbf{B}_{0:}$ $\omega^2 = \Omega_c^2 + k^2 v_s^2$ (Electrostatic ion cyclotron waves) or $\omega^2 = \omega_l^2 = \Omega_c \omega_c$ (Lower hybrid oscillations)

Electron waves (electromagnetic)

$\mathbf{B}_0 = 0 : \omega^2 = \omega_p^2 + k^2 c^2$ (Light waves)

$\mathbf{k} \perp \mathbf{B}_0, \mathbf{E}_1 // \mathbf{B}_0$: $\frac{\omega^2 k^2}{c^2} = 1 - \frac{\omega_p^2}{\omega^2}$ (O wave)

$\mathbf{k} \perp \mathbf{B}_0, \mathbf{E}_1 \perp \mathbf{B}_0$: $\frac{\omega^2 k^2}{c^2} = 1 - \frac{\omega_p^2}{\omega^2} \frac{\omega^2 - \omega_p^2}{\omega^2 - \omega_h^2}$ (X wave)

$\mathbf{k} \parallel \mathbf{B}_0 : \frac{\omega^2 k^2}{c^2} = 1 - \frac{\omega_p^2/\omega^2}{1-(\omega_c/\omega)}$ (R wave) (whistler mode)

$\frac{\omega^2 k^2}{c^2} = 1 + \frac{\omega_p^2/\omega^2}{1+(\omega_c/\omega)}$ (L wave)

Ion waves (electromagnetic)

$\mathbf{B}_0 = 0$: none

$\mathbf{k} \parallel \mathbf{B}_0 : \omega^2 = k^2 v_A^2$ (Alfven wave)

$\mathbf{k} \perp \mathbf{B}_0 : \frac{\omega^2}{k^2} = c^2 \frac{v_s^2 + v_A^2}{c^2 + v_A^2}$(Magnetosonic wave)

This set of dispersion relations is a greatly simplified one convering only the principle directions of propagation. Nonetheless, it is a very useful set of equations to have in mind as a frame of reference for discussing more complicated wave motions. It is often possible to understand a complex situation as a modification or superposition of these basic modes of oscillation.

20 The CMA Diagram (optional)

When propagation occurs at an angle to the magnetic field, the phase velocitites change with angle. Some of the modes listed above with $\mathbf{k} \parallel \mathbf{B}_0$ and $\mathbf{k} \perp \mathbf{B}_0$ change continuously into each other; other modes simply disappear at a critical angle. This complicated state of affairs is greatly clarified by Clemmow-Mullarly-Allis (CMA) diagram. The CMA diagram is valid, however, only for cold plasmas, with $T_i = T_e = 0$.Extension to finite temperatures introduces so much complexity that the diagram is no longer useful.